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Long waves generated by ground motion

By MONTGOMERY W. SLATKIN†
University of California, Los Alamos Scientific Laboratory
Los Alamos, New Mexico 87544

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The initial-value problem for waves generated by ground motion near a shore is solved using linear shallow water theory and an exponential bottom profile. It is found that long waves can be trapped along the coast and travel with the deep water wave speed, $(gh)^{\frac{1}{2}}$. The energy in these waves decays with $x^{-\frac{1}{2}}$ instead of x^{-1} so that more energy would be observed on this coast than expected on the basis of deep water wave amplitudes.

1. Introduction

In this paper, we are concerned with the problem of linear shallow water waves generated by bottom movement as might be caused by an earthquake. Several workers have considered the generation of waves in mid-ocean (see Kajiura 1963) and recently Hwang (1970) has solved the two-dimensional problem for waves generated near a shore. Here we will consider the three-dimensional problem near a shore, and, specifically, the amount of energy which goes into edge waves.

The possibility of edge waves was first discovered by Stokes (Lamb 1945), but they were considered to be unimportant. Ursell (1951), extending Stokes's analysis of a straight shore with a linear bottom profile, showed that the Stokes edge waves are the first of an infinite series of edge wave modes. Greenspan (1956) demonstrated that one or more of the edge wave modes can be excited by a pressure disturbance moving along the coast. Using observations of surges following storms along the Atlantic coast, he showed that only the lowest-order mode is excited by these storms. Greenspan and Ursell both assumed a linear bottom profile so that the water approached infinite depth far from the shore. Longuet-Higgins (1966) has shown that if the water approaches a finite depth then there are only a finite number of edge wave modes.

Munk, Snodgrass & Gilbert (1964) calculated the dispersion function ((f, n) diagram in their terminology) and compared their results with measurements made on the California continental shelf. They found that most of the energy in long waves (5–100 km wavelength) is contained in the trapped modes. They also pointed out that in the limit of long wavelength, the wave speed of the edge waves approaches $(gh_0)^{\frac{1}{2}}$ where h_0 is the depth far from the shore. This result is expected on simple physical grounds and is found to be important in the problem treated here.

† Present address: Department of Biology, University of Chicago, Chicago, Illinois 60637.

The question to be treated is how much energy goes into the different edge waves as the result of a disturbance localized in space and time. One interesting result is that if the water approaches infinite depth far from the shore then no edge waves are generated. This result follows from the fact that the speed of the edge waves generated by bottom movement is proportional to the deep water wave speed $(gh_0)^{\frac{1}{2}}$. If h_0 approaches infinity, then the edge waves do not appear in the result. This explains why Greenspan did not find any other waves generated by the moving pressure disturbance, other than the one travelling with the storm, although his 'storm' began impulsively at $t = 0$. Had he chosen a bottom profile which was finite far from the shore, then edge waves other than the one moving with the storm would have appeared. The validity of his results is not changed by this because these other waves would have been an artifact of the impulsive beginning of the disturbance.

The model here is necessarily artificial in order to get analytic results. It is most unlikely that the bottom profile used could be matched to an actual shore. However, the results of this model can illustrate the underlying simplicity of the generation of edge waves and can be used to interpret more exact numerical models. The main result is that there can be a leading edge wave generated with a wave speed of $(gh_0)^{\frac{1}{2}}$. There are also other edge waves generated with a smaller speed, but these may not be of interest because of the difference in arrival times, and will not be considered. The leading edge wave decays more slowly than the non-trapped waves which have approximately the same arrival time.

2. The model

The equation of linear shallow water theory in a non-rotating system is

$$g(\nabla . (h\nabla\zeta)) - \frac{\partial^2\zeta}{\partial t^2} = -\frac{\partial^2 h}{\partial t^2}, \tag{1}$$

where ζ is the wave amplitude and $z = -h(x, y, t)$ is the location of the bottom. If we assume that

$$h(x, y, t) = h_0(y) + h_1(x, y, t) \tag{2}$$

with $h_1 \ll h_0$, then the lowest-order equation is

$$g\frac{\partial}{\partial y}\left(h_0(y)\frac{\partial\zeta}{\partial y}\right) + gh_0\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial t^2} = -\frac{\partial^2 h_1}{\partial t^2}. \tag{3}$$

$h_0(y)$ is then the equilibrium bottom profile which we will assume is defined in $0 \leqslant y < \infty$. The shore is at $y = 0$ and the boundary condition which must be satisfied there is $(\partial\zeta/\partial y)|_{y=0} = 0$ if $h_0(0) \neq 0$ or $\zeta|_{y=0}$ finite if $h_0(0) = 0$.

We can take the Fourier transform in x and the Laplace transform in t to get

$$g\frac{d}{dy}\left(h_0(y)\frac{d\bar{\zeta}}{dy}\right) - (gh_0 k^2 + s^2)\bar{\zeta} = F(k, y, s), \tag{4}$$

where

$$\bar{\zeta}(k, y, s) = \int_{-\infty}^{\infty} e^{ikx}\int_0^{\infty} e^{-st}\zeta(x, y, t)\,dt\,dx \tag{5}$$

and

$$F(k, y, s) = -\int_{-\infty}^{\infty} e^{ikx} \int_0^{\infty} e^{-st} \frac{\partial^2 h_1}{\partial t^2} dt\, dx + s\bar{\zeta}(k, y, t = 0) + \frac{\partial \bar{\zeta}}{\partial t}(k, y, t = 0). \quad (6)$$

The method of solution does not depend on whether the waves are initiated by ground motion of finite duration or by initial surface displacement caused by impulsive ground motion. Hwang (1970) has shown that, for long waves, the effect of instantaneous ground motion is equivalent to a displacement of the surface at $t = 0$.

In the specific model considered here, the equilibrium bottom profile is

$$h_0(y) = h_0(1 - e^{-\alpha y}), \quad (7)$$

where h_0 and α are specified. This choice has the advantage of being finite at ∞ and analytic in $0 < y < \infty$, so there is no need to match solutions at any interior point. If we substitute (7) into (4) and introduce the new variable

$$u = e^{-\alpha y},$$

the resulting equation is

$$u^2(1-u)\bar{\zeta}_{uu} + u(1-2u)\bar{\zeta}_u - \left(\frac{s^2}{\alpha^2 g h_0} + \frac{k^2}{\alpha^2}(1-u)\right)\bar{\zeta} = \frac{1}{\alpha^2 g h_0} F\left(k, \frac{1}{\alpha} \ln \frac{1}{u}, s\right)$$
$$\equiv G(k, u, s). \quad (8)$$

The problem then is to solve this equation subject to the condition that ζ be finite at $u = 0$ and $u = 1$ along with the radiation condition. This is most easily done by finding the eigenfunctions to the appropriate homogeneous equation and expanding G in terms of these eigenfunctions. As usual, the eigenfunctions can be interpreted as the different possible wave modes.

3. Eigenfunctions

We want the solution to the equation

$$u^2(1-u)f'' + u(1-2u)f' - (k^2/\alpha^2)(1-u)f = -\lambda f \quad (9)$$

with f finite at $u = 0, 1$. If we look for a power series solution of the form

$$f(u) = \sum_{m=0}^{\infty} a_n u^{m+\gamma}, \quad (10)$$

then the indicial equation is

$$\gamma^2 = (k^2/\alpha^2) - \lambda \quad (11)$$

and the recursion relation is

$$a_n = \frac{(\gamma+n-1)(\gamma+n) - k^2/\alpha^2}{(\gamma+n)^2 - \gamma^2} a_{n-1}. \quad (12)$$

Assuming that k^2/α^2 is real, then if $k^2/\alpha^2 > \lambda$, only the positive root can be chosen while if $k^2/\alpha^2 < \lambda$, then both roots can be used and the solution is the sum of two series.

In the first case ($\lambda < k^2/\alpha^2$), the series defined by (12) will not converge at $u = 1$ unless it terminates. Thus, λ is restricted to values which satisfy

$$[(k^2/\alpha^2) - \lambda]^{\frac{1}{2}} = \tfrac{1}{2}\{-(2n-1) + [1 + 4k^2/\alpha^2]^{\frac{1}{2}}\}. \quad (13)$$

An additional restriction is that the right-hand side of (13) must be positive or

$$n < \tfrac{1}{2}(1+[1+4k^2/\alpha^2]^{\frac{1}{2}}) \equiv n_{\max}(k). \tag{14}$$

Therefore, there are a finite number of eigenvalues, λ_m (at least one) with polynominal eigenfunctions. The number of eigenvalues depends on k^2/α^2.

In the second case ($\lambda > k^2/\alpha^2$) the solution to (9) is the sum of two series, each of which is guaranteed to converge at $u = 0$. The solution can be made to converge at $u = 1$ only if the series are subtracted. Therefore, any $\lambda > k^2/\alpha^2$ is an eigenvalue and the eigenfunction is

$$f_\lambda(u) = \frac{1}{i}\sum_{n=0}^{\infty}(a_n u^{i[\lambda-k^2/\alpha^2]^{\frac{1}{2}}} - a_n^* u^{-i[\lambda-k^2/\alpha^2]^{\frac{1}{2}}})\, u^n, \tag{15}$$

which is real.

For the interpretation of the eigenfunctions, we must return to the y co-ordinate system. The discrete eigenfunctions f_m are all of the form

$$f_m(y) = \exp\{-[k^2/\alpha^2-\lambda_m]^{\frac{1}{2}}\alpha y\}\sum_{n=0}^{m} a_n \exp(-\alpha n y), \tag{16}$$

which are exponentially decaying at $y \to \infty$. These are the trapped modes. If we were looking for a solution to the homogeneous wave equation of the form

$$\zeta(x, y, t) = f(y)\exp(ikx - i\omega t), \tag{17}$$

then, in (9), λ would be replaced by $\omega^2/gh_0\alpha^2$ and (13) would determine the periods of the trapped waves of given wavelength. The trapped modes must satisfy

$$\omega_n^2/gh_0 < h^2$$

or

$$T_n^2 > gh_0 L^2 \tag{18}$$

where the T_n are the periods and L is the wavelength of the trapped waves. Thus, longer period waves are trapped.

For the continuous part of the spectrum, for large y, only the first term in (15) will contribute.

$$f_\lambda(y) \to \frac{1}{i}a_0(\exp\{i[\lambda-k^2/\alpha^2]^{\frac{1}{2}}\alpha y\} - \exp\{i[\lambda-k^2/\alpha^2]^{\frac{1}{2}}\alpha y\}). \tag{19}$$

Therefore,

$$\zeta \to \frac{1}{i}a_0 e^{-i\omega t}(\exp\{i(kx+[(\omega^2/gh_0)-k^2]^{\frac{1}{2}}y)\} - \exp\{i(kx-[(\omega^2/gh_0)-k^2]^{\frac{1}{2}}y)\}). \tag{20}$$

This represents the sum of two waves, each making an angle of $\theta = \omega t^{-1}$

$$k/[(\omega^2/gh_0)-k^2]^{\frac{1}{2}}$$

with the shore. Two waves are present so that the net effect is a disturbance moving along the shore. As ω^2/gh_0 increases from k^2 to infinity, θ goes from 0 to $\frac{1}{2}\pi$, so that all directions are represented. Waves of infinitely short periods are travelling perpendicular to shore.

It is easy to show that the eigenfunctions are mutually orthogonal with a weighting function $1/u$.

$$\int_0^1 \frac{1}{u} f_n(u) f_m(u)\, du = \delta_{nm} N_n, \tag{21}$$

where
$$N_n = \int_0^1 \frac{1}{u} f_n^2(u)\, du$$

and
$$\int_0^1 \frac{1}{u} f_\lambda(u) f_{\lambda'}(u)\, du = N_\lambda \delta(\lambda - \lambda'). \tag{22}$$

The second normalization factor can be found by direct integration

$$\int_0^1 \frac{du}{u} \left[\frac{1}{i} \sum_{n=0}^{\infty} (a_n u^{i[\lambda - k^2/\alpha^2]^{\frac{1}{2}}} - a_n^* \exp\{-i[\lambda - k^2/\alpha^2]^{\frac{1}{2}}\}) u^n\right]$$
$$\times \left[\frac{1}{i} \sum_{m=0}^{\infty} (a_m u^{i[\lambda' - k^2/\alpha^2]^{\frac{1}{2}}} - a_m^* \exp\{-i[\lambda' - k^2/\alpha^2]^{\frac{1}{2}}\}) u^m\right] = N_\lambda \delta(\lambda - \lambda').$$

The only term which could contribute a δ function is the one with $m = 0$, $n = 0$. In this term we can exchange variables to $u = e^{-t}$ to get

$$4a_0^2 \int_0^\infty dt \sin[\lambda' - k^2/\alpha^2]^{\frac{1}{2}} t \sin[\lambda - k^2/\alpha^2]^{\frac{1}{2}} t = 4\pi a_0^2 [\lambda - k^2/\alpha^2]^{\frac{1}{2}} \delta(\lambda - \lambda'). \tag{23}$$

If we define f_λ to have $a_0 \equiv 1$, then $N_\lambda = 4\pi[\lambda - k^2/\alpha^2]^{\frac{1}{2}}$.

A question which is less easily answered is the completeness of the eigenfunctions. We shall assume that any function which is analytic in $[0, 1]$ can be expanded in the form

$$g(u) = \sum_{n=1}^{n_{\max}} c_n f_n(u) + \int_{k^2/\alpha^2}^{\infty} c(\lambda) f_\lambda(u)\, d\lambda, \tag{24}$$

where
$$c_n = \frac{1}{N_n} \int_0^1 \frac{1}{u} f_n(u) g(u)\, du \tag{25}$$

and
$$c(\lambda) = \frac{1}{N_\lambda} \int_0^1 \frac{1}{u} f_\lambda(u) g(u)\, du.$$

This assumption is reasonable on physical grounds because all the waves of interest are included. If there were a function for which all of the c's were 0, then there could be a disturbance which would generate no waves of the type described by these eigenfunctions.

4. Forced motion

With the above analysis we can work out the details of the solution to a given problem. Here, we will consider waves initiated by ground motion of finite extent in space and time with zero total displacement and zero initial conditions. This is more complicated than the initial-value problem, but the method of solution and basic results are the same. Let us assume

$$-\frac{\partial^2 h_1}{\partial t^2} = f_1(x) f_2(y) f_3(t), \tag{26}$$

with

$$
\begin{aligned}
f_1(x) &= 1/2l_x \qquad (|x| < l_x),\\
&= 0 \qquad (|x| > l_x),\\
f_2(y) &= Ae^{-y/l_y y}\\
f_3(t) &= \cos^{2\pi t/\tau} \qquad (0 < t \leqslant \tau),\\
&= 0 \qquad (t > \tau),
\end{aligned} \tag{27}
$$

and

$$G(k,u,s) = A\frac{\sin kl_x}{k}\frac{s(1-e^{-s\tau})}{s^2+(2\pi/\tau)^2}\frac{1}{\alpha}u^{-1/\alpha l_y}\ln 1/u. \tag{28}$$

If we multiply each side of (8) by $1/(N_\lambda u)f_\lambda(u)$ (with λ one of the discrete or continuous eigenvalues) and integrate from 0 to 1, the result is

$$\left(\frac{s^2}{\alpha^2 gh_0}+\lambda\right)c_\lambda = \frac{A}{\alpha}\frac{\sin kl_x}{k}\frac{s(1-e^{-s\tau})}{s^2+(2\pi/\tau)^2}d_\lambda, \tag{29}$$

where

$$
\left.\begin{aligned}
c_\lambda &= \frac{1}{N_\lambda}\int_0^1 \frac{1}{u} f_\lambda(u)\bar{\zeta}(k,u,s)\,du,\\
d_\lambda &= \frac{1}{N_\lambda}\int_0^1 \frac{1}{u} u^{-1/l_y\alpha}\ln u f_\lambda(u)\,du.
\end{aligned}\right\} \tag{30}
$$

The integral for d_λ can be evaluated by integrating the series for ζ_λ term by term

$$
\begin{aligned}
d_m &= \frac{1}{N_m}\sum_{n=0}^{m}\frac{a_n}{(\gamma_m+n+1/l_y\alpha)^2},\\
d_\lambda &= \frac{1}{iN_\lambda}\sum_{n=0}^{\infty}\left(\frac{a_n}{(n+1/l_y\alpha)+i[(-k^2/\alpha^2)+\lambda]^{\frac{1}{2}}}-\frac{a_n^*}{(n+1/l_y)-\alpha i[\lambda-(k^2/\alpha^2)]^{\frac{1}{2}}}\right).
\end{aligned} \tag{31}
$$

From (29) and (24) we can write

$$\bar{\zeta}(k,u,s) = \frac{A}{\alpha}\frac{\sin kl_x}{k}\frac{s(1-e^{-s\tau})}{s^2+(2\pi/\tau)^2}\cdot\left(\sum_{n=1}^{n_{\max}(k)}\frac{d_n f_n(u)}{(s^2/\alpha^2 gh_0)+\lambda_n(k)}+\int_{k^2/\alpha^2}^{\infty}\frac{d_\lambda f_\lambda(u)}{(s^2/\alpha^2 gh_0)+\lambda}d\lambda\right). \tag{32}$$

Since the d's do not depend on s, it is easier to invert the Laplace transform first. We want to calculate

$$\frac{1}{2\pi i}\int_{-i\infty+c_0}^{i\infty+c_0}\frac{s(1-e^{-s\tau})}{s^2+(2\pi/\tau)^2}\frac{1}{(s^2/\alpha^2 gh_0)+\lambda}e^{st}\,ds. \tag{33}$$

There are four poles in the s plane but the contribution from the two at $\pm(2\pi/\tau)i$ will vanish for $t > \tau$. These represent the forced motion of the system and need not concern us. We are interested in the waves observed far from the source. The contribution from the poles at $\pm i\alpha(gh_0\lambda)^{\frac{1}{2}}$ is

$$\frac{1}{2}\left(\frac{(1-\exp\{-i\alpha[gh_0\lambda]^{\frac{1}{2}}\tau\})\exp\{i\alpha t[gh_0\lambda]^{\frac{1}{2}}\}}{(2\pi/\tau)^2(1/\alpha^2 gh_0)-\lambda}+\frac{(1-\exp\{i\alpha[gh_0\lambda]^{\frac{1}{2}}\tau\})\exp\{-i\alpha t[gh_0\lambda]^{\frac{1}{2}}\}}{(2\pi/\tau)^2(1/\alpha^2 gh_0)-\lambda}\right). \tag{34}$$

The first term will produce the wave travelling in the $+x$ direction and the second a wave in the $-x$ direction. It is sufficient to consider only $x > 0$.

$$\zeta^+(k,u,t) = \frac{A}{4\pi\alpha}\int_{-\infty}^{\infty} e^{-ikx}\frac{\sin l_x k}{k}$$
$$\times\left[\sum_{n=1}^{n_{\max}(k)} \frac{d_n f_n(u)}{(2\pi/\tau)^2(1/\alpha^2 gh_0)-\lambda_n}\exp\{i\alpha t[gh_0\lambda_n]^{\frac{1}{2}}\}(1-\exp\{-i\alpha\tau[gh_0\lambda_n]^{\frac{1}{2}}\})\right.$$
$$\left.+\int_{k^2/\alpha^2}^{\infty} d\lambda\,\frac{d_\lambda f_\lambda(u)}{(2\pi/\tau)^2\,(1/\alpha^2 gh_0)-\lambda}\exp\{i\alpha t[gh_0\lambda]^{\frac{1}{2}}\}(1-\exp\{i\alpha\tau[gh_0\lambda]^{\frac{1}{2}}\})\right]. \quad (35)$$

We must choose the contour in the k plane so that ζ^+ represents a wave travelling away from the source for $x > 0$. To simplify matters we will treat the summation and the integral part of (35) separately.

We want the poles and branch points of the terms in the sum. First, there will be poles when the $\lambda_n = (2\pi/\tau)^2(1/\alpha^2 gh_0)$. From (13)

$$\lambda_n = (k^2/\alpha^2)-\left(\frac{-(2n-1)+[1+4k^2/\alpha^2]^{\frac{1}{2}}}{2}\right)^2 \quad (36)$$

and there are poles on the real axis when

$$\left[\frac{k^2}{\alpha^2}-\left(\frac{2\pi}{\tau}\right)^2\frac{1}{\alpha^2 gh_0}\right]^{\frac{1}{2}} = \frac{-(2n-1)+[1+4k^2/\alpha^2]^{\frac{1}{2}}}{2} \quad (37)$$

has a real solution. This is most easily determined by a simple graphical analysis. The left-hand side of (37) is greater than the right-hand side by $(2n-1)$ for large z. From the simple character of the functions, then (37) has a real solution only if the right-hand side is positive at $k^2/\alpha^2 = (2\pi/\tau)^2(1/\alpha^2 gh_0)$. In others words, for those n which satisfy

$$n < \tfrac{1}{2}\left(1+\left[1+4\left(\frac{2\pi}{\tau}\right)^2\frac{1}{\alpha^2 gh_0}\right]^{\frac{1}{2}}\right), \quad (38)$$

the corresponding terms in the sum (35) have poles on the real k axis. There is always one such pair of poles. The exact location of these poles must be determined numerically.

The d_n all have the form

$$d_n = \sum_{j=1}^{n}\frac{a_j}{([(k^2/\alpha^2)-\lambda_m]^{\frac{1}{2}}+j+(1/l_y\alpha))^2}. \quad (39)$$

However, for real k, $[(k^2/\alpha^2)-\lambda_m^2]^{\frac{1}{2}} > 0$ so this term will contribute no poles on the real axis. Similarly, the a_i all have factors

$$\frac{1}{(\gamma+j)^2-\gamma^2}\quad (j \geqslant 1),$$

which will vanish at $\gamma = -\frac{1}{2}j$. Since γ is positive for real k, these factors also contribute no real poles. The same argument applies to the $f_m(k,u)$. Therefore, the only poles on the real axis are the solutions to (37). The condition (38) determines the number of edge wave modes which can be excited by the disturbance. There is also a branch point when $\lambda_m(k) = 0$ or when

$$k = \frac{-\alpha m(m-1)}{2m-1}. \quad (40)$$

If we choose the inversion contour to go under the poles on the real axis, then when $x > (gh_0)^{\frac{1}{2}}t$, the contour can be closed in the lower-half plane to give 0. The contributions from the poles in the lower-half plane are cancelled by similar terms from the second half of the solution, ζ^-. When $x < (gh_0)^{\frac{1}{2}}t$ is is easiest to use a steepest descent method to evaluate part of the integral. We will have to evaluate a series of integrals of the form

$$\int_{-\infty}^{\infty} \exp\{-i(kx - \alpha t[gh_0\lambda]^{\frac{1}{2}})\} \frac{\psi(k)}{(2\pi/\tau)^2[1/\alpha^2 gh_0] - \lambda_n} dk, \tag{41}$$

where $\psi(k)$ has no poles on the real k axis.

The stationary phase point for (42) is at

$$\frac{x}{[gh_0]^{\frac{1}{2}}t} = 2\frac{d}{dk}[\lambda_n(k)]^{\frac{1}{2}}$$

or

$$\frac{x}{[gh_0]^{\frac{1}{2}}t} = \frac{k}{\alpha}\frac{2n-1}{[\lambda_n(k)(1+4k^2/\alpha^2)]^{\frac{1}{2}}}. \tag{42}$$

The right-hand side is the group velocity of edge waves with wave-number k. For small k, $V_g \to [gh_0]^{\frac{1}{2}}$.

For a complete description of the solution, (42) must be evaluated for different values of x/t. However, the largest contribution will be when $\psi(k)$ has a maximum. Excluding the pole, $\psi(k)$ has a maximum at $k = 0$. This fact simplifies the result because, for small k, only the first edge wave mode ($m = 1$) need be considered. If we try to evaluate the integral with $k = 0$, some mathematical difficulties arise because the problem is degenerate at that point. To avoid this problem, we let $x = (gh_0)^{\frac{1}{2}}(t - \xi)$ with $\xi \ll 1$. Since we expect the solution to (44) to be small, we can expand in terms of k^2/α^2 to find the approximate solution. For $m = 1$

$$k_0 + \alpha[\tfrac{2}{3}\xi/t]^{\frac{1}{2}}. \tag{43}$$

The stationary phase path through $k = k_0$ is along the line $\operatorname{Re} k = \operatorname{Im} k$. If we deform the path of integration to follow this path, then the only pole included is the one on the positive real axis. The contribution from the steepest descent path is approximately

$$\exp\{ik_0[gh_0]^{\frac{1}{2}}d_1(k_0)\}u^{k_0^2/\alpha^2}\frac{\pi^{\frac{1}{2}}}{(2\pi/\tau)^2(1/\alpha^2 gh_0)(\alpha^2 gh_0\xi t)^{\frac{1}{4}}} \tag{44}$$

for $\xi(gh_0)^{\frac{1}{2}} < \tau$ and is $O(\xi^{\frac{3}{4}})$ for $\xi(gh_0)^{\frac{1}{2}} > \tau$. Because the disturbance is of finite extent, the leading edge wave is of length $(gh_0)^{\frac{1}{2}}\tau$. The contribution from the pole is

$$\exp\{i(k_r x - \alpha[gh_0\lambda_n(k_r)]^{\frac{1}{2}}t)\}\left(\frac{\psi(k_r)}{\lambda_n^1(k_r)}\right), \quad x < (gh_0)^{\frac{1}{2}}(t-\tau), \quad 0, \quad x > (gh_0)^{\frac{1}{2}}(t-\tau), \tag{45}$$

where k_r is the location of the pole. This wave represents the forced motion of the system travelling with the phase velocity which is of finite duration because of the finite length of the disturbance. Thus, this wave will not be observed far from the source.

The result is that the localized bottom movement will excite a leading edge wave travelling with wave speed $[gh_0]^{\frac{1}{2}}$. Other edge waves of smaller amplitude can be generated which will travel with different speeds. In this example these other waves would arise from the relative maxima of $\sin kl_x/k$ at $k = (m+\frac{1}{2})\pi/l_x$ and would travel with the appropriate group velocity, which is less than but proportional to $[gh_0]^{\frac{1}{2}}$. Because of the mathematical difficulties mentioned, the wave amplitude has a factor $\xi^{-\frac{1}{4}}$. This is not a serious problem because the total wave energy is finite and can be found by evaluating

$$\int_0^\tau \zeta_\xi^2 \, d\xi.$$

The integral part of (35) represents the non-trapped waves generated by the ground motion. It is easiest to estimate this integral by the limit of a sum of contributions from the different wave frequencies

$$\int_{k^2/\alpha^2}^{\infty} d\lambda \phi(\lambda, k) \exp\{i\alpha[gh_0\lambda]^{\frac{1}{2}} t\} \simeq \lim_{\substack{\Delta\lambda \to 0 \\ N\to\infty}} \sum_{j=1}^{N} \phi(\lambda_j, k) \exp\{i\alpha[gh_0\lambda_j]^{\frac{1}{2}} t\}, \qquad (46)$$

where $\lambda_j = (k^2/\alpha^2) + j\Delta\lambda$. Now each term in the sum can be inverted and the limit taken afterward.

We would like to determine whether the energy in edge waves and non-trapped waves is the same order of magnitude. If this is the case, then under some circumstances unusually large disturbances could be recorded along a coast because the edge waves would not be subject to the $1/r$ geometrical decay. For this reason, we shall only look at those waves in the continuous part of the spectrum which have a wave speed relative to the shore of nearly $[gh_0]^{\frac{1}{2}}$. If we let $x = [gh_0]^{\frac{1}{2}} t (1-\epsilon)$ then each term in the inversion of (46) is of the form

$$\int_{-\infty}^{\infty} \exp[ih(k)t]\phi(k)\, dk, \qquad (47)$$

where $$\phi(k) = \frac{\sin kl_x}{k} \frac{d_{\lambda_j}(k) f_{\lambda_j}(k)}{(2\pi/\tau)^{\frac{1}{2}}(1/\alpha^2 gh_0) - \lambda_j}$$

and $$h(k) = -[gh_0]^{\frac{1}{2}}(k(1-\epsilon) + [k^2 + j\alpha^2\Delta\lambda]^{\frac{1}{2}}).$$

The stationary point ($h'(k_s) = 0$) is at

$$k_s \simeq \alpha[j\Delta\lambda/2\epsilon]^{\frac{1}{2}}. \qquad (48)$$

There will be a significant contribution to the integral when $\phi(k_s)$ is large, i.e. when k_s is small. Therefore, only the terms in the sum with $j < (1/\alpha^2)\, 2\epsilon/\Delta\lambda$ will be important. Returning to the integral form, we have

$$\zeta^+(x, u, t) \simeq \int_0^{2\epsilon/\alpha^2} \zeta_\lambda(x, u, t)\, d\lambda, \qquad (49)$$

where ζ_λ is the result of the stationary phase integral for each term in the sum. The upper limit is somewhat arbitrary for the approximate calculation, but it must be $O(\epsilon)$. From the above analysis, we have

$$\zeta_\lambda^+([gh_0]^{\frac{1}{2}} t(l-\epsilon), u, t) \simeq \psi_s(k_s) \exp[i(h(k_s) - \tfrac{1}{4}\pi)] \,.\, [\pi/\epsilon^3[gh_0\lambda]^{\frac{1}{2}} t]^{\frac{1}{2}}. \qquad (50)$$

In order to compare (49) and (44) (the edge wave with velocity $[gh_0]^{\frac{1}{2}}$ we must choose $0 < \epsilon < \tau/t$. For large t, $\epsilon \ll 1$ and (49) is given approximately by

$$\zeta \simeq \zeta_{\epsilon/\alpha^2}(2\epsilon/\alpha^2)$$

or

$$\zeta \simeq \frac{A \sin l_x/2^{\frac{1}{2}}}{l_x/2^{\frac{1}{2}}} \frac{dy_{2-\frac{1}{2}} f y_{2-\frac{1}{2}}(u)}{(2\pi/\tau)^2(1/\alpha^2 g h_0)} \left[\frac{\pi}{[gh_0]^{\frac{1}{2}}}\right]^{\frac{1}{2}} \frac{2}{t^{\frac{1}{2}}}. \quad (51)$$

As expected, the energy in this type of wave decreases with $1/t$ as opposed to $1/t^{\frac{1}{2}}$ for the leading edge wave. Since the coefficients of (44) and (51) are of the same order of magnitude, we can expect that in some cases a significant fraction of energy can be put into a leading wave travelling with the deep water wave speed.

5. Discussion

The details of the above calculations have not been worked out because their purpose is to illustrate the possible importance of edge waves generated by explosions or earthquakes near a coast, not to provide a model for experimental results. Too many simplifying assumptions have been made for agreement with experiments to be anything more than accidental. The important result is that a leading edge wave travels with the deep water wave speed and might not be distinguishable from the non-trapped waves by arrival time. This is a possible explanation for the unusually large response of the harbour in Crescent City, California to the tsunami generated by the 1964 Alaskan earthquake.

This result would also lend support to Carrier's (1970) hypothesis that energy from a tsunami could be trapped or guided by an undersea ridge. Although the problem solved here is quite different, the result would suggest that some of the trapped waves on the ridge could also travel with the deep water wave speed. If observations stations are placed on or near such a ridge, they would indicate a larger deep water wave amplitude than stations off the ridge. As Carrier has pointed out, this could furnish a partial explanation for unusually large wave runups which have been measured.

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